ROLLING WITHOUT SLIPPING OF SPHERICAL CURVE PAIRS WITH COMMON GEODESIC NORMALS

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ABSTRACT

This study investigates the kinematic and differential geometric properties of curve pairs on the unit sphere in three-dimensional Euclidean space that share a common geodesic normal direction. In particular, curvature relations between such curves undergoing rolling without slipping are analyzed. The transformations between their Sabban frames are derived, and the corresponding rotation angles associated with the rolling motion are characterized. Furthermore, the geodesic curvatures of the curves and the geometric relationships between them are examined.
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INTRODUCTION

One of the fundamental research areas in differential geometry is the study of the geometric and topological properties of curves on the unit sphere. In this context, the differential geometric analysis of spherical curves has been enriched through concepts such as geodesic curvature, torsion, and total torsion, providing a detailed characterization of the local and global behavior of curves in space. Moreover, the topological structure of spherical curves, including connectedness, homeomorphism, and the characterization of components, has also occupied a significant place in the literature. In particular, the investigation of the number of connected components of the space formed by closed curves on the 2-sphere when their geodesic curvatures lie within a prescribed interval represents a notable contribution to this field (Saldanha & Zühlke, 2013).

The differential geometry of spherical motions holds significant importance not only from a theoretical standpoint but also in terms of mechanical and kinematic applications. Expressing rigid body motions via Euler parameters and relating these motions to points on a unit-radius hypersphere clearly elucidates the connection between the motion and the behavior of spherical curves (McCarthy, 1987).

Studies aimed at understanding the behavior of spherical curves within more general geometric structures have also received considerable attention in the literature. The introduction of the concept of geodesic curvature for horizontal curves on three-dimensional contact sub-Riemannian manifolds has enabled the quantification of these curves' deviation from true geodesics (Barilari & Kohli, 2022). Furthermore, it has been demonstrated that geodesic spherical curves in Euclidean, hyperbolic, and spherical spaces can be characterized by simple linear equations through the use of rotation-minimizing (RM) frames. Such characterizations reveal that a Riemannian manifold having constant sectional curvature is equivalent to the condition that the corresponding curves satisfy specific linear relationships (Silva & Silva, 2020). Additionally, in three-dimensional manifolds, the requirement that the total torsion of closed spherical curves vanishes emerges as a distinctive feature associated with constant sectional curvature.

The behavior of curves on the three-dimensional sphere with constant geodesic curvature and torsion is also noteworthy. Such curves can be expressed as the superposition of circular motions occurring in two orthogonal planes, and their global behavior is either periodic or dense on a Clifford torus, distinguishing them from classical helices in Euclidean space (Chakrabarti et al., 2019). Similarly, the well-known result that the total torsion of closed spherical curves vanishes has been generalized to three-dimensional Riemannian manifolds with constant curvature, thereby providing a deeper understanding of the relationship between differential geometric properties and topological structure (Pansonato & Costa, 2008).

The study of curves on the unit sphere holds significant importance not only from a theoretical mathematical perspective but also across various applied domains. Spherical curves have found diverse applications in disciplines such as physics, computer science, mechanics, and data science. For instance, the periodic trajectories of charged particles on spherical surfaces under magnetic fields are associated with closed curves possessing specific geodesic curvature functions (Schneider, 2011). Similarly, spherical curves serve as important tools in transformation optics (Horsley, 2011). In computer-aided geometric design, they are effectively employed in areas such as optimal placement problems, surface decomposition, and biological modeling (Behandish & Ilieş, 2016; Tang et al., 1998; Waxman, 2006). Within the context of data science, spherical curves play a crucial role in shape analysis of human motion and in dimensionality reduction techniques (Devanne et al., 2015; Lazar & Lin, 2017). Collectively, these studies highlight both the local and global properties of spherical curves, thereby fostering interdisciplinary applications and interactions.

The kinematic differential geometry of curves under spatial motion provides an effective framework for analyzing the movement of curves from a geometric perspective. Within this context, concepts such as linear motions, axes of rotation, curvature, and torsion play a central role. Examples of studies in this area include magnetic curves and Lorentz forces derived from the spherical indicatrices of ruled surfaces (Yıldırım & Kasap, 2024), as well as the relationships between Mannheim partner curves of spherical curves under one-parameter helical motions (Orbay & Şahin, 2022). Additionally, investigations into pairs of curves rolling without slipping on the unit sphere and the relationships between their Mannheim partner curves have yielded significant results (Orbay, 2025).

In this study, the kinematic and differential geometric properties of curve pairs sharing geodesic normal directions on the unit sphere in three-dimensional Euclidean space are investigated. In particular, the curvature relationships between these curve pairs, which roll over each other without slipping, are examined. To this end, the transformation relations between the Sabban frames of the curves and the corresponding rotation angles associated with the rolling motion are analyzed. Furthermore, the geodesic curvatures of the curves and the geometric relationships among them are interpreted. This approach provides a fundamental tool for understanding the kinematic behavior of curve pairs with common geodesic normal directions and offers a novel perspective that complements existing studies in literature.

METHODOLOGY

In spherical geometry, curves on the sphere exhibit distinct geometric behavior compared to curves in the Euclidean plane due to the presence of the Sabban frame, as introduced by Sabban (Koenderink, 1990). This distinction directly influences both the local and global geometric properties of the curves (Izumiya & Nagai, 2017). The curves considered in the present study, along with their associated Sabban frames, are defined as follows:

Let (α) and (β) be two curves on the unit sphere S^2 that roll over each other without slipping. The associated curves sharing their geodesic normal directions are denoted by $(\bar{\alpha})$ and $(\bar{\beta})$, respectively. The parameters $s, \bar{s}, r, \bar{r}$ correspond to the arc lengths of the curves $(\alpha), (\bar{\alpha}), (\beta)$, and $(\bar{\beta})$, respectively. The corresponding points of the curve pairs with common geodesic normal directions $((\alpha), (\bar{\alpha}))$ and $((\beta), (\bar{\beta}))$ on S^2 are expressed as

$$\alpha(s) \leftrightarrow \bar{\alpha}(\bar{s}) \text{ and } \beta(r) \leftrightarrow \bar{\beta}(\bar{r}).$$

The Sabban frames associated with these curves are given by

$$\{\alpha, T^{\alpha}, B^{\alpha}\}, \{\bar{\alpha}, T^{\bar{\alpha}}, B^{\bar{\alpha}}\}, \{\beta, T^{\beta}, B^{\beta}\}, \{\bar{\beta}, T^{\bar{\beta}}, B^{\bar{\beta}}\}.$$

Here, $\alpha, \beta, \bar{\alpha}, \bar{\beta}$ represent the position vectors, while $T^{\alpha}, T^{\bar{\alpha}}, T^{\beta}, T^{\bar{\beta}}$ denote the corresponding unit tangent vectors. The geodesic normal vectors are defined as

$$B^{\alpha} = \alpha \times T^{\alpha}, B^{\bar{\alpha}} = \bar{\alpha} \times T^{\bar{\alpha}}, B^{\beta} = \beta \times T^{\beta}, B^{\bar{\beta}} = \bar{\beta} \times T^{\bar{\beta}}.$$

The derivative equations of the Sabban frames are given as follows:

$$\alpha_s = T^{\alpha}, T_s^{\alpha} = -\alpha + \gamma^{\alpha} B^{\alpha}, B_s^{\alpha} = -\gamma^{\alpha} T^{\alpha} \quad (1)$$

$$\bar{\alpha}_{\bar{s}} = T^{\bar{\alpha}}, T_{\bar{s}}^{\bar{\alpha}} = -\bar{\alpha} + \gamma^{\bar{\alpha}} B^{\bar{\alpha}}, B_{\bar{s}}^{\bar{\alpha}} = -\gamma^{\bar{\alpha}} T^{\bar{\alpha}} \quad (2)$$

$$\beta_r = T^{\beta}, T_r^{\beta} = -\beta + \gamma^{\beta} B^{\beta}, B_r^{\beta} = -\gamma^{\beta} T^{\beta} \quad (3)$$

$$\bar{\beta}_{\bar{r}} = T^{\bar{\beta}}, T_{\bar{r}}^{\bar{\beta}} = -\bar{\beta} + \gamma^{\bar{\beta}} B^{\bar{\beta}}, B_{\bar{r}}^{\bar{\beta}} = -\gamma^{\bar{\beta}} T^{\bar{\beta}}. \quad (4)$$

Here, the functions $\gamma^{\alpha}, \gamma^{\bar{\alpha}}, \gamma^{\beta}, \gamma^{\bar{\beta}}$ represent the geodesic curvatures of the corresponding curves (Ravani & Ku, 1991).

RESULTS AND DISCUSSION

In spherical geometry, the geodesic normal vector in the Sabban frame of a curve is one of the fundamental components that determines the curve's orientation on the surface as well as its curvature dependent geometric behavior. Therefore, the coincidence of geodesic normal directions at corresponding points of two curves indicates the existence of a strong geometric relationship between them. Curve pairs sharing common geodesic normal directions also have particular significance from a kinematic perspective. In particular, the rolling motion defined between such curves without slipping requires the surface-adapted orientations at the points of contact to be compatible. This condition necessitates that the rolling motion be considered within a more constrained and therefore more structured geometric framework compared to classical Euclidean contact problems.

In this section, the differential geometric relationships arising under the rolling without slipping condition for the curve pairs under consideration will be examined in detail. The relations between the geodesic curvatures of the curves will be derived and interpreted with the aid of the components of the Sabban frame and the associated rotation angles.

For the curve pairs $((\alpha), (\bar{\alpha}))$ and $((\beta), (\bar{\beta}))$, which share common geodesic normal directions, the following relations hold between their Sabban frames:

$$\begin{aligned} \bar{\alpha} &= (\cos\theta)\alpha + (\sin\theta)T^{\alpha} \\ T^{\bar{\alpha}} &= (-\sin\theta)\alpha + (\cos\theta)T^{\alpha} \\ B^{\bar{\alpha}} &= B^{\alpha} \end{aligned} \quad (5)$$

and

$$\bar{\beta} = (\cos\psi)\beta + (\sin\psi)T^{\beta}$$
$$T^{\bar{\beta}} = (-\sin\psi)\beta + (\cos\psi)T^{\beta} \qquad (6)$$
$$B^{\bar{\beta}} = B^{\beta}$$

Here, the angles $\theta = \theta(\bar{s})$ and $\psi = \psi(\bar{r})$ denote the angles between the position vectors of α and $\bar{\alpha}$, and β and $\bar{\beta}$, respectively.

The relationship between the Sabban frames of the curves (α) and (β), which roll over each other without slipping, is given as follows:

$$\beta = \alpha$$
$$T^{\beta} = (\cos\phi)T^{\alpha} + (\sin\phi)B^{\alpha} \qquad (7)$$
$$B^{\beta} = (-\sin\phi)T^{\alpha} + (\cos\phi)B^{\alpha}$$

Here, the angle $\phi = \phi(r)$ denotes the angle between the unit tangent vectors T^{α} and T^{β} . If $\phi = 0$, the curves are tangent in the same direction, and their frames coincide. The quantity $\alpha_r = \frac{d\alpha}{dr}$ represents the rotational velocity of the geodesic Frenet frame in the tangent plane during the rolling without slipping (Struik, 1988).

Corollary 1. While the curves (α) and (β) roll over each other without slipping on the unit sphere, the relationship between the Sabban frames of the curves (α) and $(\bar{\beta})$ is obtained as follows:

$$\bar{\beta} = (\cos\psi)\alpha + (\sin\psi)(\cos\phi)T^{\alpha} + (\sin\psi)(\sin\phi)B^{\alpha}$$
$$T^{\bar{\beta}} = (-\sin\psi)\alpha + (\cos\psi)(\cos\phi)T^{\alpha} + (\cos\psi)(\sin\phi)B^{\alpha} \qquad (8)$$
$$B^{\bar{\beta}} = (-\sin\phi)T^{\alpha} + (\cos\phi)B^{\alpha}$$

Corollary 2. While the curves (α) and (β) roll over each other without slipping on the unit sphere, the relationship between the Sabban frames of the curves $(\bar{\alpha})$ and (β) is obtained as follows:

$$\bar{\alpha} = (\cos\theta)\beta + (\sin\theta)(\cos\phi)T^{\beta} + (\sin\theta)(-\sin\phi)B^{\beta}$$
$$T^{\bar{\alpha}} = (-\sin\theta)\beta + (\cos\theta)(\cos\phi)T^{\beta} + (\cos\theta)(-\sin\phi)B^{\beta} \qquad (9)$$
$$B^{\bar{\alpha}} = (\sin\phi)T^{\beta} + (\cos\phi)B^{\beta}$$

Corollary 3. While the curves (α) and (β) roll over each other without slipping on the unit sphere, the relationship between the Sabban frames of the curves $(\bar{\alpha})$ and $(\bar{\beta})$ is obtained as follows:

$$\bar{\alpha} = [(\cos\theta)(\cos\psi) + (\sin\theta)(\cos\phi)(\sin\psi)]\bar{\beta}$$
$$+[(\cos\theta)(-\sin\psi) + (\sin\theta)(\cos\phi)(\cos\psi)]T^{\bar{\beta}} + (\sin\theta)(-\sin\phi)B^{\bar{\beta}} \qquad (10)$$
$$T^{\bar{\alpha}} = [(-\sin\theta)(\cos\psi) + (\cos\theta)(\cos\phi)(\sin\psi)]\bar{\beta}$$

$$+[(sin\theta)(sin\psi)+(cos\theta)(cos\phi)(cos\psi)]T^{\overline{\beta}}+(cos\theta)(-sin\phi)B^{\overline{\beta}}$$

$$B^{\overline{\alpha}}=(sin\phi)(sin\psi)\overline{\beta}+(sin\phi)(cos\psi)T^{\overline{\beta}}+(cos\phi)B^{\overline{\beta}}$$

Theorem 1. During the rolling without slipping motion of two curves (α) and (β) on the unit sphere, both these curves and their associated curves sharing common geodesic normal directions are great circles on the sphere.

Proof**.** Differentiating the relation $\bar{\alpha}=(cos\theta)\alpha+(sin\theta)T^{\alpha}$ given in equation (5) with respect to $\bar{s}$, we obtain

$$\bar{\alpha}_{\bar{s}}=(-sin\theta)\theta_{\bar{s}}\alpha+(cos\theta)\alpha_{\bar{s}}+(cos\theta)\theta_{\bar{s}}T^{\alpha}+(sin\theta)T^{\alpha}_{\bar{s}}.$$

Using equation (2) and the change of parameter in differentiation yields

$$T^{\bar{\alpha}}=(-sin\theta)\theta_{\bar{s}}\alpha+(cos\theta)\alpha_{s}s_{\bar{s}}+(cos\theta)\theta_{\bar{s}}T^{\alpha}+(sin\theta)T^{\alpha}_{s}s_{\bar{s}}.$$

Taking into account equations (1) and (5), we obtain

$$(-sin\theta)\alpha+(cos\theta)T^{\alpha}=(-sin\theta)\theta_{\bar{s}}\alpha+(cos\theta)T^{\alpha}s_{\bar{s}}+(cos\theta)\theta_{\bar{s}}T^{\alpha}$$
$$+(sin\theta)(-\alpha+\gamma^{\alpha}B^{\alpha})s_{\bar{s}},$$

and

$$(-sin\theta)\alpha+(cos\theta)T^{\alpha}=(-sin\theta)(\theta_{\bar{s}}+s_{\bar{s}})\alpha+(cos\theta)(\theta_{\bar{s}}+s_{\bar{s}})T^{\alpha}+(sin\theta)\gamma^{\alpha}s_{\bar{s}}B^{\alpha}.$$

So,

$$(-sin\theta)=(-sin\theta)(\theta_{\bar{s}}+s_{\bar{s}}),\quad (cos\theta)=(cos\theta)(\theta_{\bar{s}}+s_{\bar{s}}),\quad (sin\theta)\gamma^{\alpha}s_{\bar{s}}=0.$$

From this, it follows that

$$\theta_{\bar{s}}+s_{\bar{s}}=1,\ \ (sin\theta)\gamma^{\alpha}s_{\bar{s}}=0. \tag{21}$$

Since the curves are parametrized by arc length$s_{\bar{s}}\neq 0$. Moreover, $sin\theta\neq 0$, as otherwise the curves would coincide or be antipodal, which corresponds to a trivial case. Hence,

$$\gamma^{\alpha}=0 \tag{32}$$

This implies that the curve (α) is a great circle.

Now, differentiating $T^{\bar{\alpha}}=(-sin\theta)\alpha+(cos\theta)T^{\alpha}$ with respect to $\bar{s}$, we obtain

$$T^{\bar{\alpha}}_{\bar{s}}=(-cos\theta)\theta_{\bar{s}}\alpha+(-sin\theta)\alpha_{\bar{s}}+(-sin\theta)\theta_{\bar{s}}T^{\alpha}+(cos\theta)T^{\alpha}_{\bar{s}}.$$

Using equations (1), (2), and (5), we obtain

$$-\bar{\alpha}+\gamma^{\bar{\alpha}}B^{\bar{\alpha}}=(-cos\theta)\theta_{\bar{s}}\alpha+(-sin\theta)\alpha_{s}s_{\bar{s}}+(-sin\theta)\theta_{\bar{s}}T^{\alpha}+(cos\theta)T^{\alpha}_{s}s_{\bar{s}},$$
$$-\bar{\alpha}+\gamma^{\bar{\alpha}}B^{\bar{\alpha}}=(-cos\theta)(\theta_{\bar{s}}+s_{\bar{s}})\alpha+(-sin\theta)(\theta_{\bar{s}}+s_{\bar{s}})T^{\alpha}+(cos\theta)\gamma^{\alpha}s_{\bar{s}}B^{\alpha},$$
$$(-cos\theta)\alpha+(-sin\theta)T^{\alpha}+\gamma^{\bar{\alpha}}B^{\alpha}=(-cos\theta)(\theta_{\bar{s}}+s_{\bar{s}})\alpha+(-sin\theta)(\theta_{\bar{s}}+s_{\bar{s}})T^{\alpha}$$
$$+(cos\theta)\gamma^{\alpha}s_{\bar{s}}B^{\alpha}.$$

From this

$$\theta_{\bar{s}}+s_{\bar{s}}=1,\ \ \gamma^{\bar{\alpha}}=(cos\theta)\gamma^{\alpha}s_{\bar{s}}. \tag{43}$$

Since $cos\theta\neq 0$ and from (12), it follows that

$$\gamma^{\bar{\alpha}} = 0. \qquad (54)$$

Next, consider equation (6). Differentiating $\bar{\beta} = (cos\psi)\beta + (sin\psi)T^{\beta}$ with respect to $\bar{r}$, we obtain

$$\bar{\beta}_{\bar{r}} = (-sin\psi)\psi_{\bar{r}}\beta + (cos\psi)\beta_{\bar{r}} + (cos\psi)\psi_{\bar{r}}T^{\beta} + (sin\psi)T^{\beta}_{\bar{r}}.$$

Using equations (3), (4), and (6), we obtain

$$(-sin\psi)\beta + (cos\psi)T^{\beta} = (-sin\psi)(\psi_{\bar{r}} + r_{\bar{r}})\beta + (cos\psi)(\psi_{\bar{r}} + r_{\bar{r}})T^{\beta} + (sin\psi)r_{\bar{r}}\gamma^{\beta}B^{\beta}.$$

By comparing coefficients, it follows that

$$\psi_{\bar{r}} + r_{\bar{r}} = 1, \ \ (sin\psi)\gamma^{\beta}r_{\bar{r}} = 0. \qquad (65)$$

Since $r_{\bar{r}} \neq 0$ and $sin\psi \neq 0$, it follows that

$$\gamma^{\beta} = 0. \qquad (76)$$

Next, differentiate $T^{\bar{\beta}} = (-sin\psi)\beta + (cos\psi)T^{\beta}$ with respect to $\bar{r}$. Employing again equations (3), (4), and (6), we obtain

$$T^{\bar{\beta}}_{\bar{r}} = (-cos\psi)\psi_{\bar{r}}\beta + (-sin\psi)\beta_{\bar{r}} + (-sin\psi)\psi_{\bar{r}}T^{\beta} + (cos\psi)T^{\beta}_{\bar{r}},$$

$$(-cos\psi)\beta + (-sin\psi)T^{\beta} + \gamma^{\bar{\beta}}B^{\beta} = (-cos\psi)(\psi_{\bar{r}} + r_{\bar{r}})\beta + (-sin\psi)(\psi_{\bar{r}} + r_{\bar{r}})T^{\beta}$$
$$+(cos\psi)r_{\bar{r}}\gamma^{\beta}B^{\beta},$$

$$\psi_{\bar{r}} + r_{\bar{r}} = 1, \gamma^{\bar{\beta}} = (cos\psi)\gamma^{\beta}r_{\bar{r}}. \qquad (87)$$

Since $r_{\bar{r}} \neq 0$ and $cos\psi \neq 0$, and using (16), it follows that

$$\gamma^{\bar{\beta}} = 0. \qquad (98)$$

Consequently, during the rolling without slipping motion of the curves (α) and (β) on the unit sphere, both these curves and their associated curves sharing common geodesic normal directions are great circles on the sphere.

Theorem 2. Let (α) and (β) be two curves rolling over each other without slipping on the unit sphere. Then, the following relation holds between their geodesic curvatures:

$$\gamma^{\beta} = \phi_r + \gamma^{\alpha}(sec\phi).$$

Proof. Differentiating the relation $T^{\beta} = (cos\phi)T^{\alpha} + (sin\phi)B^{\alpha}$ given in equation (7) with respect to r, we obtain

$$T^{\beta}_r = (-sin\phi)\phi_r T^{\alpha} + (cos\phi)T^{\alpha}_r + (cos\phi)\phi_r B^{\alpha} + (sin\phi)B^{\alpha}_r.$$

Taking into account equations (1), (3), and (7), we have

$$-\beta + \gamma^{\beta}B^{\beta} = (-sin\phi)\phi_r T^{\alpha} + (cos\phi)T^{\alpha}_s s_r + (cos\phi)\phi_r B^{\alpha} + (sin\phi)B^{\alpha}_s s_r,$$

$$-\beta + \gamma^{\beta}B^{\beta} = (-sin\phi)\phi_r T^{\alpha} + (cos\phi)(-\alpha + \gamma^{\alpha}B^{\alpha})s_r + (cos\phi)\phi_r B^{\alpha} + (sin\phi)(-\gamma^{\alpha}T^{\alpha})s_r.$$

Rewriting, we obtain

$$-\alpha + \gamma^{\beta}[(-sin\phi)T^{\alpha} + (cos\phi)B^{\alpha}] = (-cos\phi)s_r\alpha + (-sin\phi)(\phi_r + \gamma^{\alpha}s_r)T^{\alpha}$$
$$+(cos\phi)(\phi_r + \gamma^{\alpha}s_r)B^{\alpha}.$$

By comparing coefficients, it follows that

$$(cos\phi)s_r = 1, \gamma^{\beta} = \phi_r + \gamma^{\alpha}s_r. \quad (109)$$

Hence,

$$\gamma^{\beta} = \phi_r + \gamma^{\alpha}(sec\phi).$$

This theorem shows that, for two curves rolling over each other without slipping on the unit sphere, the relationship between their geodesic curvatures depends on the angle between their tangent vectors.

Considering equations (17) and (19) together, the following theorem is obtained.

Theorem 3. Let (α) ve (β) be two curves on the unit sphere rolling over each other without slipping and let $((\alpha),(\bar{\alpha}))$ ve $((\beta),(\bar{\beta}))$ denote pairs of curves sharing common geodesic normal directions. Then, the geodesic curvature of $\bar{\beta}$satisfies

$$\gamma^{\bar{\beta}} = (cos\psi)(\phi_{\bar{r}} + \gamma^{\alpha}s_{\bar{r}}).$$

Geometrically, On the unit sphere, during the rolling without slipping of two curves over each other, the geodesic curvature of the curve with which the geodesic normal direction of one of the curves coincides is determined by the geodesic curvature of the other curve and the change in the rolling angle of the initial curve. This result demonstrates a direct and intrinsic relationship between the local differential geometry of a curve and the kinematic constraints imposed by rolling without slipping on the unit sphere. Consequently, the curvature of one curve governs the geometric behavior of its rolling partner, providing a rigorous characterization of their interdependent motions on spherical surfaces.

Example 1. On the unit sphere, consider the curves

$$\alpha(s) = (cos\, s, sin\, s, 0), \beta(r) = (\cos r, 0, \sin r).$$

which are both great circles. Their corresponding unit tangent and geodesic normal vectors are

$$T^{\alpha}(s) = (-\sin s, \cos s, 0), B^{\alpha}(s) = (0,0,1),$$
$$T^{\beta}(r) = (-\sin r, 0, cos\, r), B^{\beta}(r) = (0,-1,0).$$

The associated curves with coinciding geodesic normals, obtained from equations (5) and (6), are given by

$$\bar{\alpha}(s) = (cos(\theta + s), sin(\theta + s), 0), \bar{\beta}(r) = (cos(\psi + r), 0, sin(\psi + r)).$$

In this configuration, all curves are great circles, and the stated theorems are satisfied. For the parameter ranges $s, r \in [0,2\pi], \theta = \frac{\pi}{6}, \psi = \frac{\pi}{4}$ the curves are depicted in Figure 1.

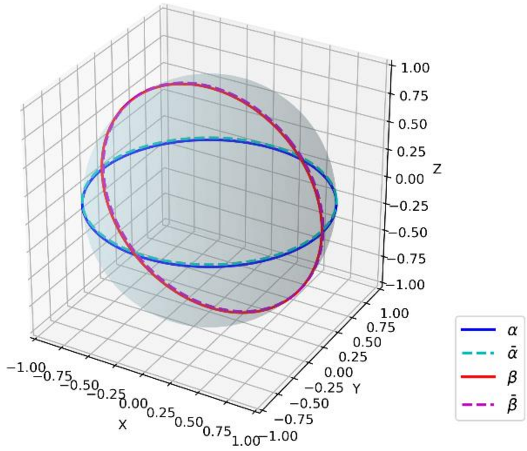

Figure 1. The curve pairs $((\alpha),(\bar{\alpha}))$ and $((\beta),(\bar{\beta}))$ are great circle pairs.

Example 2. On the unit sphere, consider the curves parameterized by arc length:

$$\alpha(s) = (\cos k \cos\left(\frac{s}{\cos k}\right), \cos k \sin\left(\frac{s}{\cos k}\right), \sin k),$$

$$\beta(s) = (\cos\varphi \cos k \cos\left(\frac{s}{\cos k}\right) - \sin\varphi \sin\left(\frac{s}{\cos k}\right), \cos\varphi \cos k \sin\left(\frac{s}{\cos k}\right) + \sin\varphi \cos\left(\frac{s}{\cos k}\right), \cos\varphi \sin k)$$

,

where $k, \varphi \in \left(0, \frac{\pi}{2}\right)$ are fixed constants. The corresponding tangent and geodesic normal vectors of the Sabban frame for these curves are:

$$T^{\alpha}(s) = (-\sin\left(\frac{s}{\cos k}\right), \cos\left(\frac{s}{\cos k}\right), 0),$$

$$B^{\alpha}(s) = (-\cos\left(\frac{s}{\cos k}\right)\sin k, -\sin\left(\frac{s}{\cos k}\right)\sin k, \cos k),$$

$$T^{\beta}(s) = (-\cos\phi \sin\left(\frac{s}{\cos k}\right) - \sin\phi \sin k \cos\left(\frac{s}{\cos k}\right),$$

$$\cos\phi \cos\left(\frac{s}{\cos k}\right) - \sin\phi \sin k \sin\left(\frac{s}{\cos k}\right), \sin\phi \cos k),$$

$$B^{\beta}(s) = (\sin\phi \sin\left(\frac{s}{\cos k}\right) - \cos\phi \sin k \cos\left(\frac{s}{\cos k}\right),$$

$$-\sin\phi \cos\left(\frac{s}{\cos k}\right) - \cos\phi \sin k \sin\left(\frac{s}{\cos k}\right), \cos\phi \cos k).$$

The associated curves with coinciding geodesic normals, obtained from equations (5) and (6), are:

$$\bar{\alpha}(s) = \left(\cos\theta \cos k \cos\left(\frac{s}{\cos k}\right) - \sin\theta \sin\left(\frac{s}{\cos k}\right),\right.$$

$$\left.\cos\theta \cos k \sin\left(\frac{s}{\cos k}\right) + \sin\theta \cos\left(\frac{s}{\cos k}\right), \cos\theta \sin k\right),$$

$$\bar{\beta}(s) = \left(\cos\psi \cos\varphi \cos k \cos\left(\frac{s}{\cos k}\right) - \cos\psi \ \sin\varphi \sin\left(\frac{s}{\cos k}\right) - \sin\psi \cos\phi \sin\left(\frac{s}{\cos k}\right)\right.$$

$$- \sin\psi \sin\varphi \sin k \cos\left(\frac{s}{\cos k}\right),$$

$$\cos\psi \cos\varphi \cos k \sin\left(\frac{s}{\cos k}\right) + \cos\psi \sin\varphi \cos\left(\frac{s}{\cos k}\right)$$

$$+ \sin\psi \cos\phi \cos\left(\frac{s}{\cos k}\right) - \sin\psi \sin\phi \sin k \sin\left(\frac{s}{\cos k}\right),$$

$$\left.\cos\psi \cos\varphi \sin k + \sin\psi \sin\phi \cos k\right).$$

For the parameter range $s \in [0, 2\pi \cos k]$ and the values $k = \frac{\pi}{6}, \theta = \frac{\pi}{6}, \psi = \frac{\pi}{5}, \varphi = \frac{\pi}{4}$, the curves are illustrated in Figure 2.

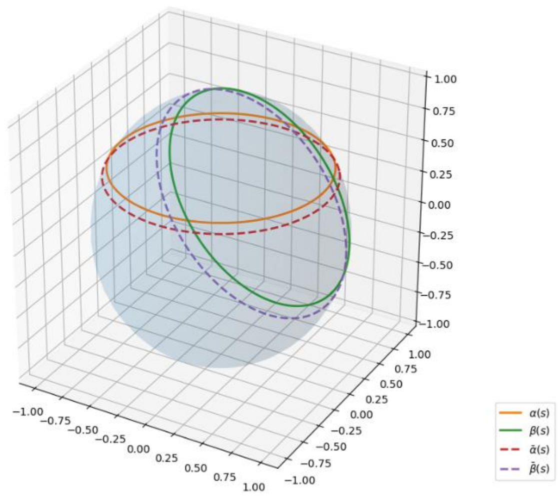

Figure 2. The curve pairs $((\alpha),(\bar{\alpha}))$ and $((\beta),(\bar{\beta}))$ are circle pairs.

CONCLUSION

In this study, the kinematic and differential geometric properties of curve pairs on the unit sphere S^2 sharing common geodesic normal directions have been systematically investigated. The analysis has been carried out within the framework of the rolling without slipping, which provides a natural geometric setting to relate the intrinsic properties of the curves.

By employing the Sabban frame, explicit transformation relations between corresponding curve pairs have been established. These relations allowed us to derive fundamental connections between the geodesic curvatures of the curves in terms of the rolling angles and their variations. In particular, it has been shown that under the rolling without slipping condition, both the curves and their associated curves with coinciding geodesic normals are necessarily great circles. This result reveals a strong geometric restriction imposed by the kinematic constraints. Furthermore, a key outcome of this work is the explicit relation between the geodesic curvatures of the rolling curves, demonstrating that the curvature of one curve is governed by the curvature of the other together with the rate of change of the rolling angle. This establishes a direct link between the local differential geometry and the underlying motion. The illustrative examples presented in the paper confirm the theoretical findings and provide geometric insight into the behavior of such curve pairs on the unit sphere. These examples also highlight how abstract relations can be visualized through concrete parametrizations.

Overall, the results contribute to a deeper understanding of the interplay between differential geometry and kinematics on spherical surfaces. The framework developed in this study not only extends existing results in literature but also opens new directions for further research, particularly in the study of constrained motions, spherical kinematics, and their applications in geometric modeling and related fields.

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